An FPGA-specific Approach to Floating-Point Accumulation and Sum-of-Products

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Outline

Context of this work

Proposed accumulator

Improved sum-of-products

Conclusions

Context of this work

Summing a large number of **floating-point** *terms fast and accurately*

Crucial for:

- **Scientific computations:**
  - dot-product, matrix-vector product, matrix-matrix product
  - numerical integration
- **Financial simulations:**
  - Monte-Carlo simulations
- ...

Not familiar with floating-point arithmetic?

Floating-Point(FP) numbers

Let x be a **normalized** binary FP number:

$$x = (-1)^S \times 1.f \times 2^e$$

where:

S - the **sign** of x

f - the **fraction** of x.

e - the **exponent** of x

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- e gives the dynamic range
  - IEEE-754 FP **double precision**, e_{min}=-1022 and $e_{max} = 1023$
- number of bits of f gives the **precision** p
  - IEEE-754 FP **double precision**, p=52

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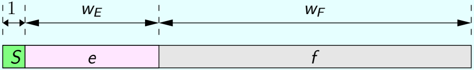
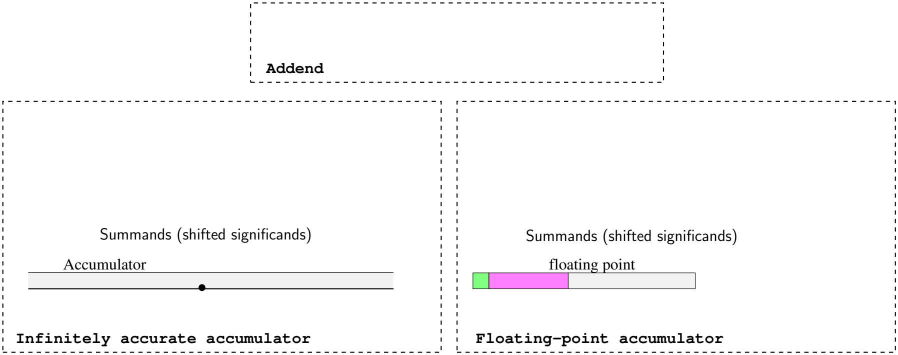
Graphical representation:



Accumulation



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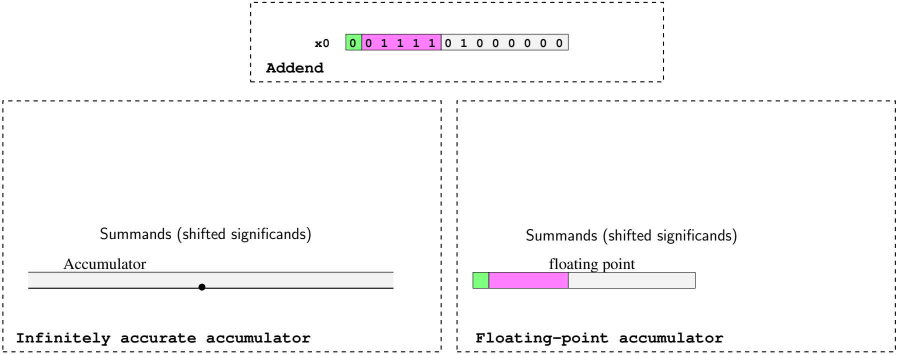


Accumulation

x0 0 0 1 1 1 1 0 1 0 0 0 0 0 0

Addend

x0 1 0 1 0 0 0 0 0 0

Summands (shifted significands)

Accumulator

1.0 1 0 0 0 0 0 0

Infinitely accurate accumulator

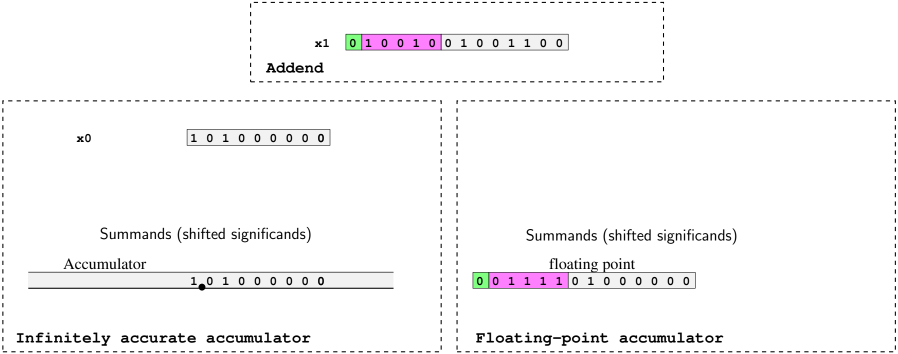
Summands (shifted significands)

floating point

0 0 1 1 1 1 0 1 0 0 0 0 0 0

Floating-point accumulator

Accumulation



Accumulation

x1 `0 1 0 0 1 0 0 1 0 0 1 1 0 0`

Addend

x0 `1 0 1 0 0 0 0 0 0`
x1 `1 0 1 0 0 1 1 0 0`

Summands (shifted significands)

Accumulator
`1 0 1 1.1 0 1 0 0 0 0 0`

Infinitely accurate accumulator

x1 `1 0 1 0 0 1 1 0 0`
acc `1 0 1 0 0 0 0 0 0`

Summands (shifted significands)

floating point
`0 1 0 0 1 0 0 1 1 1 0 1 0 0`

Floating-point accumulator

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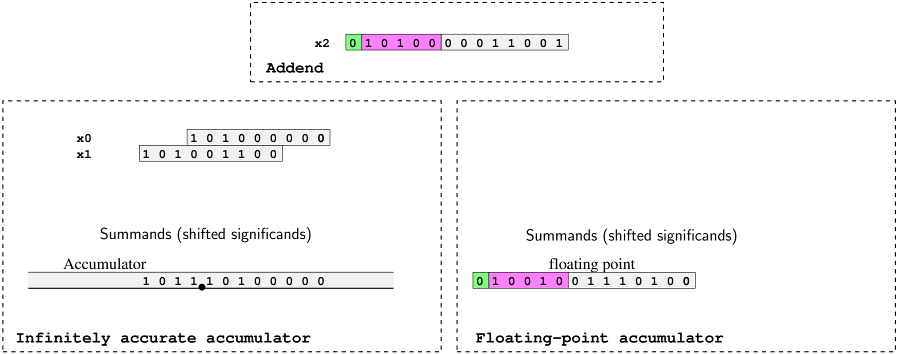

Accumulation

x2 0 1 0 1 0 0 0 0 0 1 1 0 0 1

Addend

```
x0            1 0 1 0 0 0 0 0 0
x1      1 0 1 0 0 1 1 0 0
x2  1 0 0 0 1 1 0 0 1
```

Summands (shifted significands)

Accumulator

1 0 1 1 1 0.1 1 0 0 0 0 0 0

Infinitely accurate accumulator

```
x2   1 0 0 0 1 1 0 0 1
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Summands (shifted significands)

floating point

0 1 0 1 0 0 0 1 1 1 0 1 1 0

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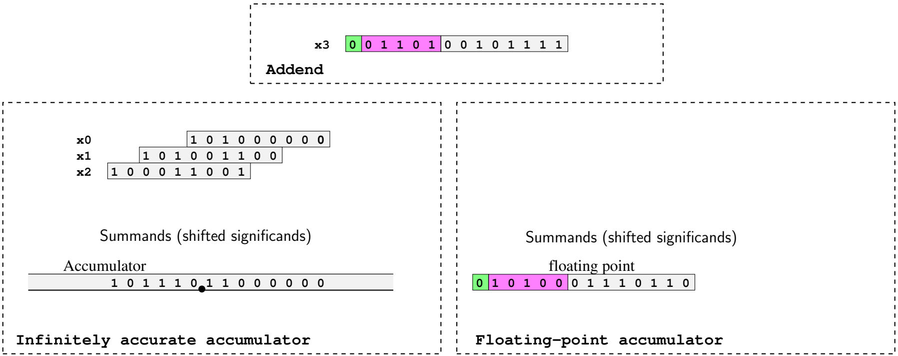

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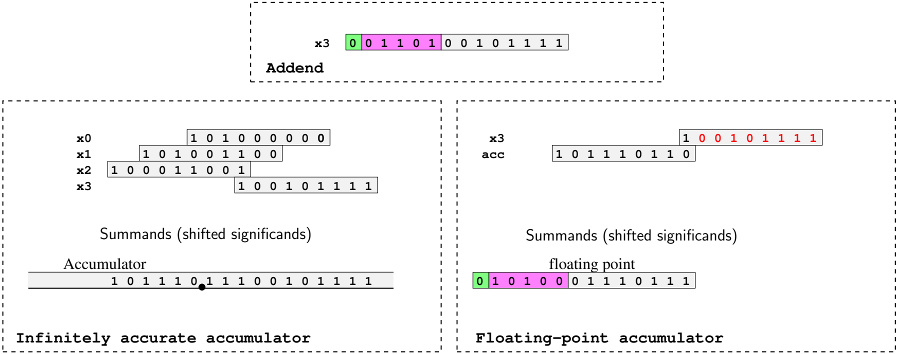

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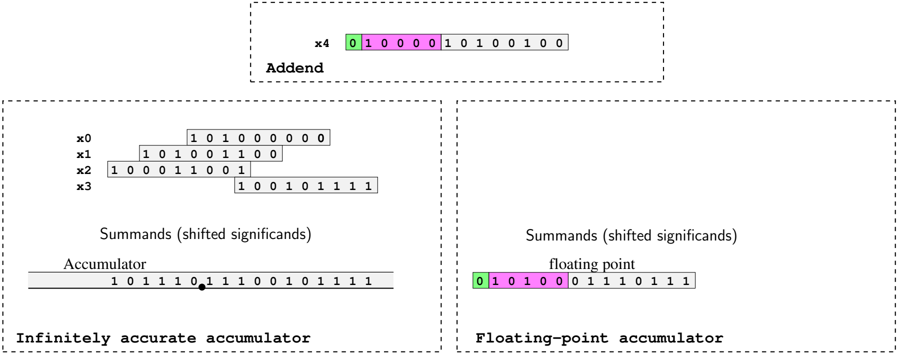

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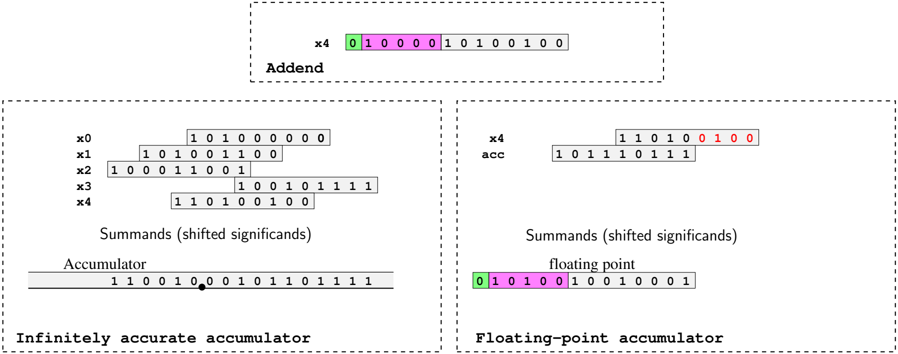

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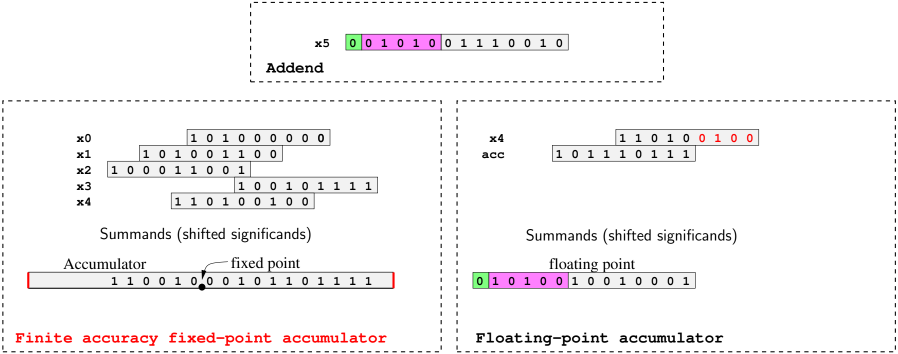

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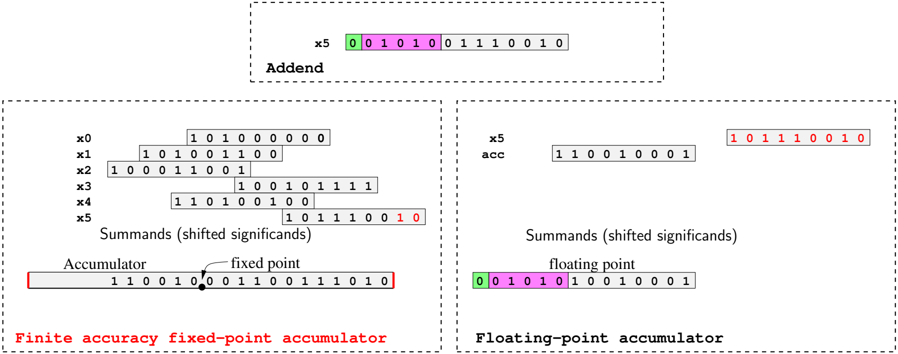

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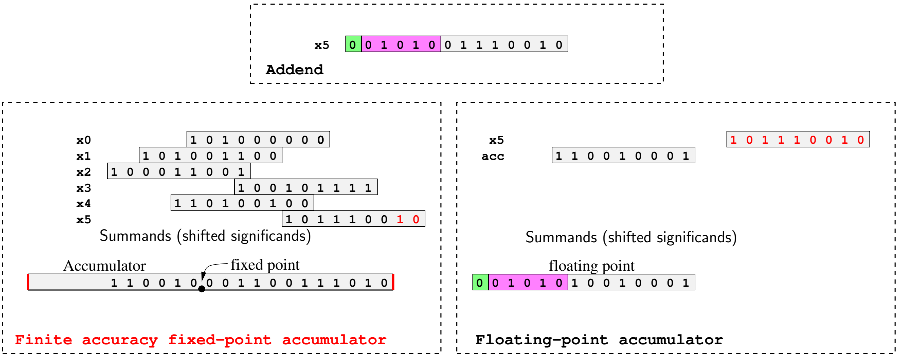
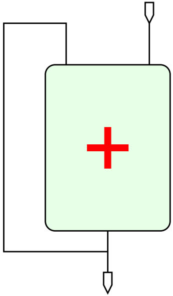


Accuracy:

Exact Result	=	50.2017822265625
FP Acc	=	50.125
Fixed-Point Acc	=	50.20166015625

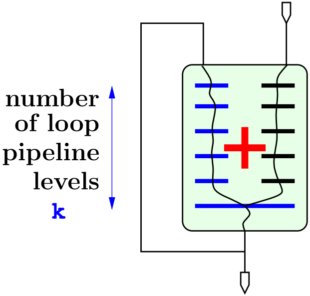
Closer look



Accumulator based on combinatorial floating-point adder

- very low frequency
- must pipeline for larger frequency

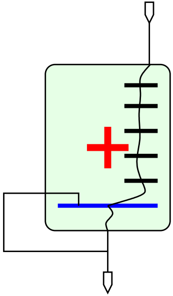
Closer look



Accumulator based on pipelined floating-point adder

- loop's critical path contains 2 shifters
- shifters are deeply pipelined
- produces k accumulation results
- these results have to be added somehow
  - adder tree
  - multiplexing mechanism on accumulation loop

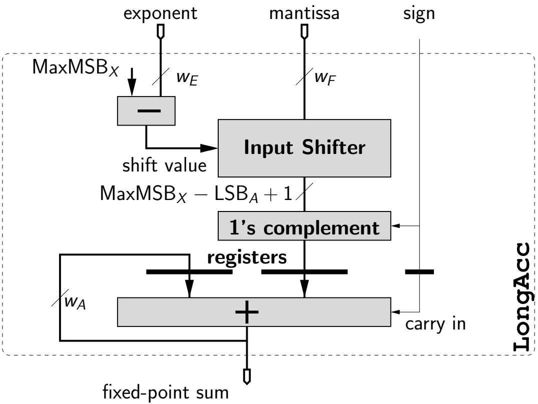
Closer look



Accumulator based on proposed long accumulator

- no shifts on the loop's critical path
- returns the result of the accumulation in fixed point
- the alignment shifter pipeline depth does not concern the result

Accumulator Architecture



- the sum is kept as a **large fixed-point number**
- one **alignment shift** (size depends on $MaxMSB_X$ and LSB_A)
- the loop's **critical path** contains a **fixed-point addition**
- fixed-point addition is fast on current FPGAs

Fast Accumulator Design

The accumulator should run at a target frequency

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- 64-bit addition works at 220MHz on Xilinx Virtex4 FPGA due to fast-carry chains
- still not enough ?
- use *partial carry-save representation*
  - cut large carry-propagation into chunks of k bits
  - critical path $=$ k-bit addition
  - small cost: $\lfloor width_{accumulator}/k \rfloor$ registers
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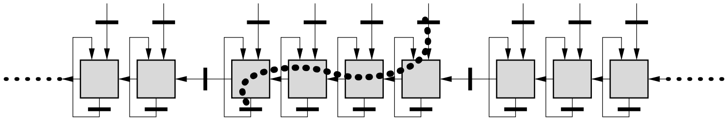

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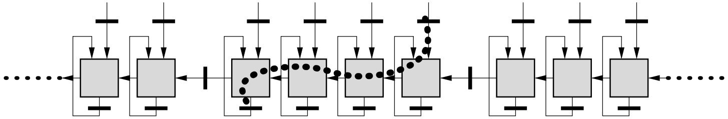

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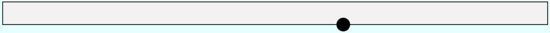
We advocate:

An **application tailored** fixed-point accumulator
for **floating-point inputs**

Ensuring that:

1. accumulator significand never needs to be shifted
2. it never overflows
3. provides a **result as accurate as the application requires**

Accumulator Parameters



MSB_A the weight of the MSB of the accumulator

- must to be larger than max. expected result

$MaxMSB_X$ the max. weight of the MSB of the summand

LSB_A weight of the LSB of the accumulator

- determines the final accumulation accuracy

The designer must provide values for these parameters.

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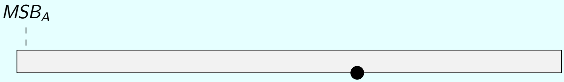

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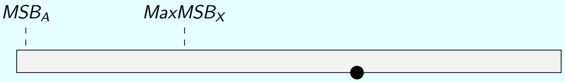

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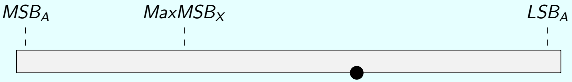

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Two possibilities:

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How to chose the parameters using the rough error analysis ?

MSB_A
- know an actual maximum + 10 bits safety margin
- consider the number of terms to sum

$MaxMSB_X$
- exploit input properties + safety margin
- worst case: $MaxMSB_X = MSB_A$

LSB_A **precision** vs. **performance**
- consider the desired final precision
- sum n terms, at most $\log_2 n$ bits are invalid

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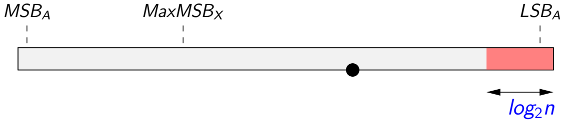
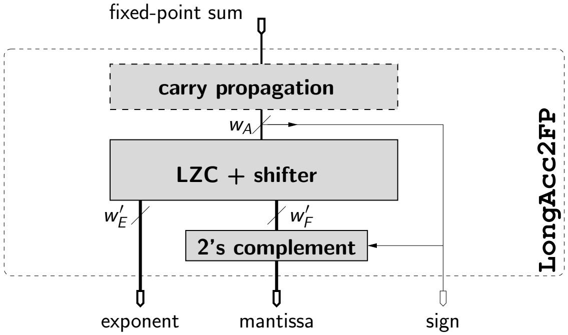
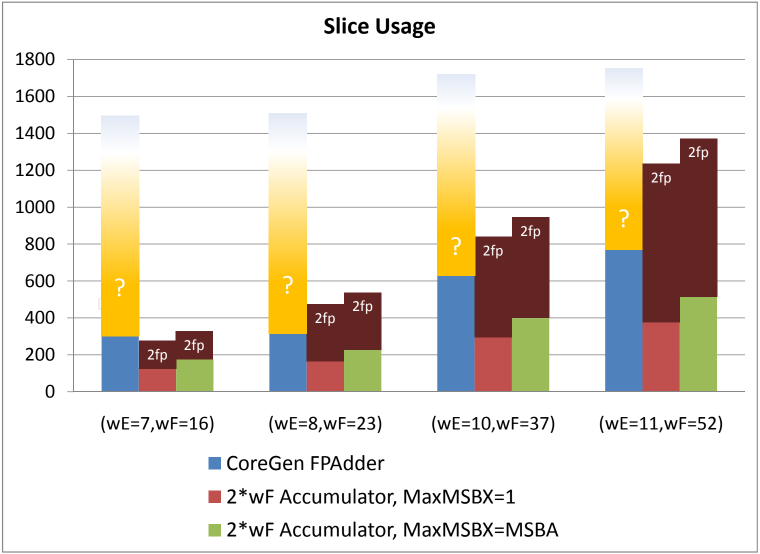


Post-normalization unit, or not



- converts fixed-point accumulator format to floating-point
- pipelined unit may be shared by several accumulators
- less useful:
  - many applications do not need the running sum
  - better to do conversion in software, use FPGA to accelerate the computation

Performance results



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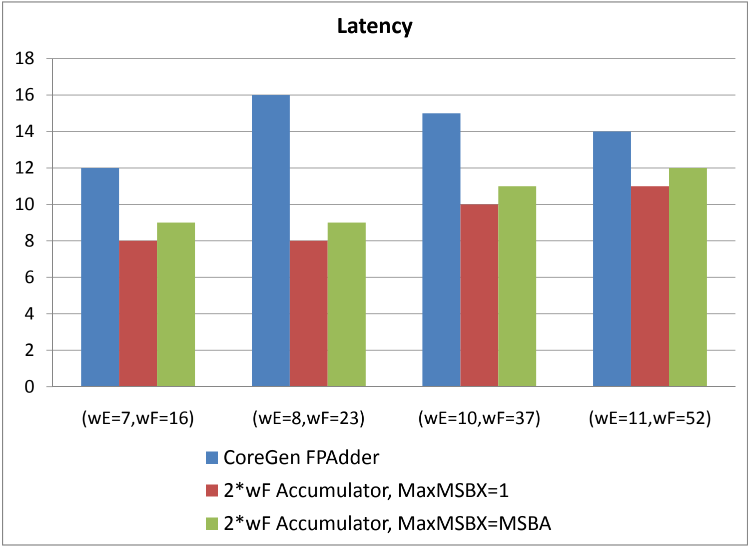
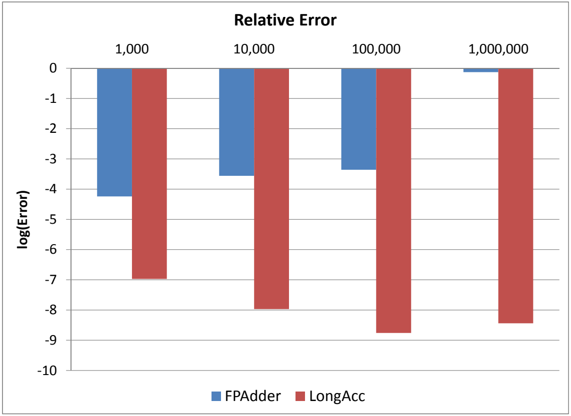


Relative error results



Accumulation of FP($w_E = 7, w_F = 16$) in unif. [0,1]

- LongAcc ($MSB_A = 20,\ LSB_A = -11$)

Accurate Sum-of-Products

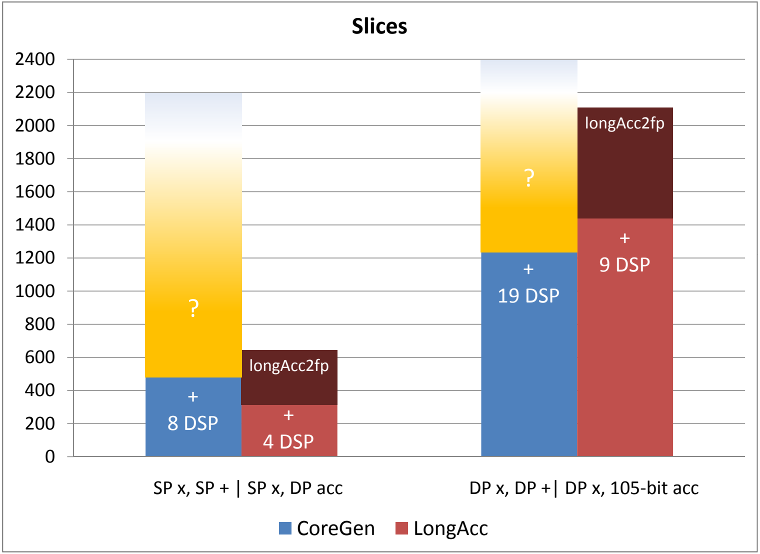
Ideea

Accumulate **exact** results of all multiplications

1. Use exact multipliers:
   - return all the bits of the exact product
   - contain no rounding logic
   - are cheaper to build
2. Feed the accumulator with exact multiplication results

Cost: Input shifter of accumulator is twice as large

Operator Performance



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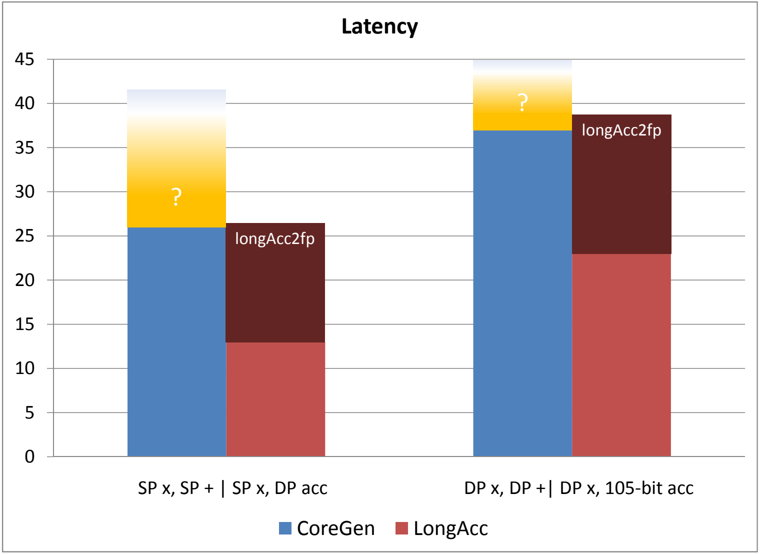
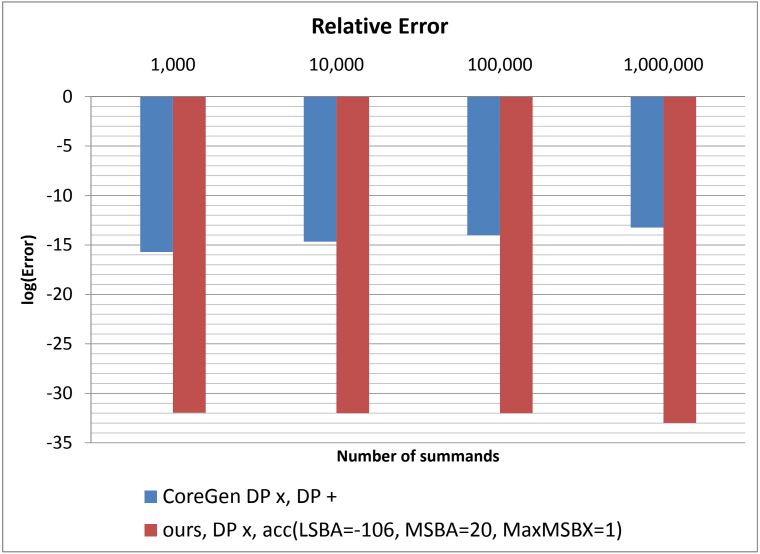


Operator Accuracy



Conclusion

- floating-point on FPGA should use the flexibility of the FPGA, not reimplement operators available in microprocessors
- faster and arbitrarily more accurate than a naive floating-point approach
- cost: designer-provided bounds on the accumulated values
- reward: improved performance + provably accurate accumulation
- available under GPL at: `http://www.ens-lyon.fr/LIP/Arenaire/Ware/FloPoCo/`

Thank you for your attention !

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Questions ?

Absolute and Relative Error

- measuring **only absolute error is not enough**
- the $error_{absolute} = |x - \tilde{x}|$
  - **example 1**
    x=182341298371283 12.19238712398 7

    $\tilde{x}$=18234129837128312.19238712398 6

    $error_{absolute} = 1e - 12$
  - **example 2**
    x=0.19238712398 7

    $\tilde{x}$=0.19238712398 6

    $error_{absolute} = 1e - 12$
- better measure **relative error** (percentage error)
- $error_{rel} = \frac{\tilde{x} - x}{x}$
  - **example 1:** $error_{rel} = -5e - 29$
  - **example 2:** $error_{rel} = -5e - 12$

Absolute and Relative Error

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    $error_{absolute} = 1e - 12$
- better measure **relative error** (percentage error)
- $error_{rel} = \frac{\tilde{x} - x}{x}$
  - **example 1:** $error_{rel} = -5e - 29$
  - **example 2:** $error_{rel} = -5e - 12$

Absolute and Relative Error

- measuring **only absolute error is not enough**
- the $error_{absolute} = |x - \tilde{x}|$
  - **example 1**
    x=1823412983712831 2.192387123987
    $\tilde{x}$=18234129837128312.192387123986
    $error_{absolute} = 1e - 12$
  - **example 2**
    x=0.192387123987
    $\tilde{x}$=0.192387123986
    $error_{absolute} = 1e - 12$
- better measure **relative error** (percentage error)
- $error_{rel} = \frac{\tilde{x} - x}{x}$
  - **example 1:** $error_{rel} = -5e - 29$
  - **example 2:** $error_{rel} = -5e - 12$